Geometrical Characteristics of Flat-Faced Bodies of Revolution

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Transition curves between flat faces and parallel middle bodies are developed as families of special polynomials termed "cubic" polynomials. The curves start and end with zero curvature to provide no discontinuities in curvature at the junction with the flat faces and the parallel middle bodies. Each "cubic" polynomial is a linear combination of independent polynomials controlled by adjustable parameters. Permissible ranges of the adjustable parameters are examined with respect to selected geometrical constraints such as inflection points.

Nomenclature

a_i = polynomial coefficients
C = constant of integration
D = diameter of parallel middle body
D_f = diameter of flat face
F = integral defined in Eq. (31)
h = "cubic" polynomial
$K_1[x]$ = polynomial corresponding to k_1
$\bar{K}_1[x]$ = polynomial corresponding to $\bar{k}_1$
$\bar{K}_0[x]$ = polynomial corresponding to $\bar{k}_0$
k = curvature
$\bar{k}$ = rate of change of curvature with arc length
$\bar{k}_1$ = rate of change of curvature with arc length at $x = 1$
$\bar{k}_0$ = rate of change of curvature with arc length at $x = 0$
$Q[x]$ = polynomial for restraining conditions
s = arc length
X = axial coordinate
X_n = axial length of forebody
x = normalized axial coordinate
Y = radius
y = normalized radius
α = unspecified constant
α_i = adjustable conditions
β = unspecified constant
β_j = restraining conditions
γ = unspecified constant
$(\ )'$ = single differentiation with x
$(\ )''$ = double differentiation with x
$(\ )'''$ = triple differentiation with x

Introduction

THE geometrical characteristics of streamlined bodies of revolution are developed in Ref. 1 by the method of independent polynomials and adjustable parameters. For the case of flat-faced forebodies, a cubic polynomial is introduced for the transition curve from the flat face to the section with maximum diameter. The cubic polynomial has the desirable property of providing infinite slope and zero curvature at the junction with the flat face. The cubic polynomial of Ref. 1 has one adjustable parameter k_1, the curvature at the junction with the section of maximum diameter.

It is now desired to develop cubic polynomials that fair into parallel middle bodies, that is, having zero curvature at the junction with the parallel middle body. For the cubic polynomial of Ref. 1, k_1 is zero to satisfy the noncurvature of the parallel middle body and the resulting polynomial has no adjustable parameter. It is now proposed to develop cubic polynomials to fit parallel middle bodies so that the polynomials have adjustable parameters. Two adjustable parameters are considered: 1) $\bar{k}_0$, the rate of change of curvature with arc length at the junction with the flat face; 2) $\bar{k}_1$, the rate of change of curvature with arc length at the junction with the parallel middle body.

The polynomials are subjected to geometrical considerations such as those concerning inflection points to provide suitable ranges of values for the adjustable parameters.

The pressure distribution on a body in a flow is governed by the curvature of the body shape, among other factors. Discontinuities or sharp changes in curvature are in general to be avoided since they lead to pressure fluctuations which may have undesirable results such as separation or cavitation. Such changes in curvature may occur at junctions of bodies or at inflection points. Hence curvatures at junctions should be matched and inflection points avoided in most cases.

An earlier study of elliptic curves bridging the flat face and the parallel middle body is given in Ref. 2.

Two-Parameter "Cubic" Polynomials

The cubic polynomial for the transition curves of forebodies between flat-faced noses and cylindrical parallel middle bodies requires a zero curvature at the junctions. For the cubic polynomial of Ref. 1, $k_1 = 0$ and

$$y^3 = Q[x] = 1 + (x - 1)^3 \tag{1}$$

This may be termed the zero-parameter cubic polynomial. The normalized coordinates x and y are given by (Fig. 1),

$$x = X/X_n$$
$$y = (2Y - D_f)/(D - D_f)$$

where X is axial coordinate starting from the flat face, Y is radial distance from centerline, X_n is axial length of forebody, D is diameter of parallel middle body, and D_f is diameter of flat face

To provide a more general family of bodies, two additional adjustable parameters are introduced. Since curvature k represents the rate of change of slope with arc length, the rate of change of curvature with arc length $\bar{k}$ ought to be a useful parameter. The two adjustable parameters are then $\bar{k}_0$ and $\bar{k}_1$, $\bar{k}$ at $x = 0$ and at $x = 1$, respectively.

The rate of change of curvature with arc length is given by dk/ds. Curvature k may be written for convenience in the form

$$k = (d^2x/dy^2)[1 + (dx/dy)^2]^{-3/2} \tag{2}$$

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which yields

$$\tilde{k} \equiv \frac{dk}{ds} = \left(\frac{dk}{dy}\right)\left(\frac{dy}{ds}\right) = \left(\frac{d^3x}{dy^3}\right)\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{-2} - 3\left(\frac{dx}{dy}\right)\left(\frac{d^2x}{dy^2}\right)^2\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{-3} \quad (3)$$

At $x = 0$, we require

$$dx/dy = d^2x/dy^2 = 0$$

and then

$$\tilde{k}_0 = (d^3x/dy^3)_{x=0} \quad (4)$$

Curvature k may be written now in the alternative form

$$k = (d^2y/dx^2)[1 + (dy/dx)^2]^{-3/2} \quad (5)$$

(where x is taken as the independent variable since $dy/dx = 0$ at $x = 1$), which yields

$$\tilde{k} \equiv \frac{dk}{ds} = \frac{dk}{dx}\frac{dx}{ds} = \left(\frac{d^3y}{dx^3}\right)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-2} - 3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-3} \quad (6)$$

At $x = 1$, we require $dy/dx = d^2y/dx^2 = 0$ and then

$$\tilde{k}_1 = (d^3x/dy^3)_{x=1} \quad (7)$$

To obtain the two-parameter cubic polynomials, the method of Ref. 1 is followed.

There are two conditions, α_1 and α_2, for the two adjustable parameters

$$\alpha_1: \ \tilde{k}_0 = (d^3y/dx^3)_{x=0}$$

$$\alpha_2: \ \tilde{k}_1 = (d^3y/dx^3)_{x=1}$$

The boundary conditions β_j are

$$\beta_1: \ x = 0, \ y = 0$$
$$\beta_2: \ x = 1, \ y = 1$$
$$\beta_3: \ x = 1, \ dy/dx = 0$$
$$\beta_4: \ x = 1, \ d^2y/dx^2 = 0$$

Since there are six conditions in all, the required polynomial is of the fifth degree.

The α_i and β_j are substituted into the cubic polynomial

$$y^3 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \quad (8)$$

to give for each condition

$$\alpha_1: \ a_1 = 6/\tilde{k}_0$$
$$\alpha_2: \ 2a_3 + 8a_4 + 20a_5 = \tilde{k}_1$$
$$\beta_1: \ a_0 = 0$$
$$\beta_2: \ a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1$$
$$\beta_3: \ a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0$$
$$\beta_4: \ 2a_2 + 6a_3 + 12a_4 + 20a_5 = 0$$

The form of the cubic polynomial is then

$$y^3 = (1/\tilde{k}_0)\tilde{K}_0[x] + \tilde{k}_1\tilde{K}_1[x] + Q[x] \quad (9)$$

where $\tilde{K}_0$, $\tilde{K}_1$ and Q are fifth-degree polynomials which are determined as follows:

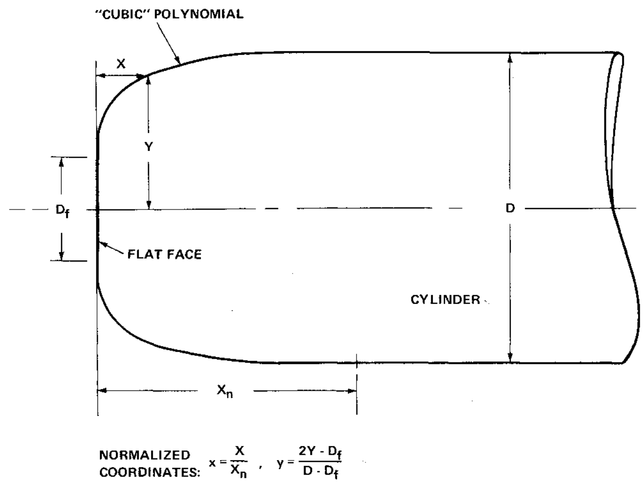

Fig. 1 Basic geometry of flat-faced nose.

The preceding relations for α_i and β_j correspond to conditions on y^3, the cubic polynomial, instead of y, as follows:

$$\alpha_1: \ (dy^3/dx)[0] = a_1 = 6/\tilde{k}_0$$
$$\alpha_2: \ (dy^3/dx)[1] = 6a_3 + 24a_4 + 60a_5 = 3\tilde{k}_1$$
$$\beta_1: \ y^3[0] = a_0 = 0$$
$$\beta_2: \ y^3[1] = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1$$
$$\beta_3: \ (dy^3/dx)[1] = a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0$$
$$\beta_4: \ (d^2y^3/dx^2)[1] = 2a_2 + 6a_3 + 12a_4 + 20a_5 = 0$$

In terms of the polynomials $\tilde{K}_0[x]$, $\tilde{K}_1[x]$ and $Q[x]$ the preceding relations represent

$$\alpha_1: \ \tilde{K}_0'[0] = 6, \ \tilde{K}_1'[0] = Q'[0] = 0$$
$$\alpha_2: \ \tilde{K}_1'''[1] = 3, \ \tilde{K}_0'''[1] = Q'''[1] = 0$$
$$\beta_1: \ \tilde{K}_0[0] = \tilde{K}_1[0] = Q[0] = 0$$
$$\beta_2: \ \tilde{K}_0[1] = \tilde{K}_1[1] = 0, \ Q[1] = 1$$
$$\beta_3: \ \tilde{K}_0'[1] = \tilde{K}_1'[1] = Q'[1] = 0$$
$$\beta_4: \ \tilde{K}_0''[1] = \tilde{K}_1''[1] = Q''[1] = 0$$

Here $\tilde{K}_0' = d\tilde{K}_0/dx$, $\tilde{K}_0'' = d^2\tilde{K}_0/dx^2$, etc.

Evaluation of $\tilde{K}_0$

Since $\tilde{K}_0[0] = \tilde{K}_0[1] = \tilde{K}_0'[1] = \tilde{K}_0''[\mp] = \tilde{K}_0'''[1] = 0$,

$$\tilde{K}_0 = \alpha x(x-1)^4 \quad (10)$$

Also since $\tilde{K}'[0] = 6$, $\alpha = 6$. Then

$$\tilde{K}_0 = 6x(x-1)^4 \quad (11)$$

Evaluation of $\tilde{K}_1$

Since $\tilde{K}_1[0] = \tilde{K}_1'[0] = \tilde{K}_1[1] = \tilde{K}_1'[1] = \tilde{K}_1''[1] = 0$,

$$\tilde{K}_1 = \beta x^2(x-1)^3 \quad (12)$$

Also, since $\tilde{K}'''[1] = 3$, $\beta = 1/2$. Then

$$\tilde{K}_1 = (1/2)x^2(x-1)^3 \quad (13)$$

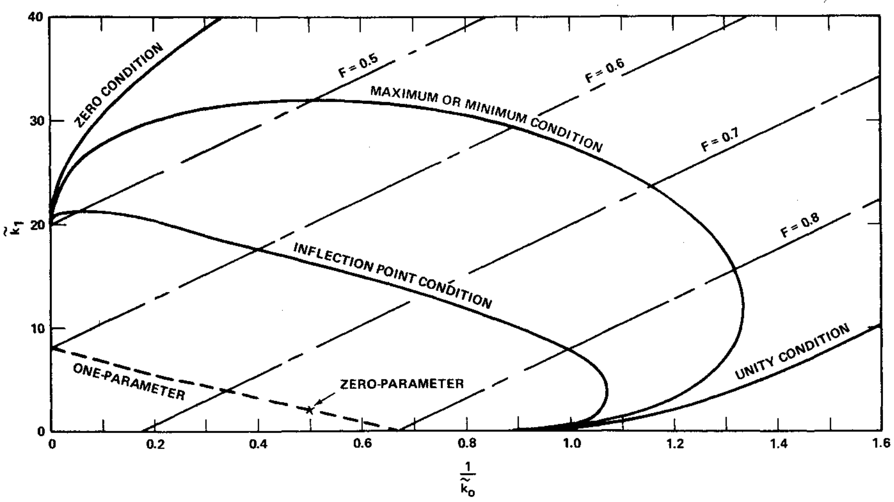

Fig. 2 Two parameter cubic polynomial-permissible ranges of $1/\tilde{k}_0$ and $\tilde{k}_1$.

Evaluation of Q

Since $Q'[0] = Q'[1] = Q''[1] = Q'''[1] = 0$,

$$Q' = \gamma x(x-1)^3 \tag{14}$$

and

$$Q = \gamma(x^5/5 - 3x^4/4 + x^3 - x^2/2) + C \tag{15}$$

From $Q[0] = 0$, $C = 0$ and from $Q[1] = 1$, $\gamma = -20$. Then

$$Q = 1 - (x-1)^4(4x+1) \tag{16}$$

Permissible Ranges of $\tilde{k}_0$ and $\tilde{k}_1$

Not all combinations of $\tilde{k}_0$ and $\tilde{k}_1$ give desirable shapes. It is interesting to analyze possible limitations in terms of simple criteria. These are 1) zero condition: $y = 0$, negative values of y are meaningless; 2) unity condition: $y = 1$, bulges above $y = 1$ are undesirable; 3) maximum or minimum condition: $dy/dx = 0$, maxima or minima other than at $x = 1$ are undesirable; 4) inflection point condition: $d^2y/dx^2 = 0$.

Inflection points are considered undesirable on noses of bodies. As explained in Ref. 1, an envelope curve may be determined for each of the preceding conditions. As shown in Fig. 2, desirable values of $\tilde{k}_0$ and $\tilde{k}_1$ are on the inside of the envelope curve.

1. Zero condition

$$y^3 = h[x;\ (1/\tilde{k}_0),\ \tilde{k}_1] = 0 \quad 0 \leq x \leq 1 \tag{17}$$

The envelope in $1/\tilde{k}_0$ and $\tilde{k}_1$ with x as the variable parameter is given by

$$h' = \partial h/\partial x = 0 \tag{18}$$

The two envelope conditions, Eqs. (17) and (18) provide two simultaneous equations in $1/\tilde{k}_0$ and $\tilde{k}_1$ which are solved by the Cramer rule to give

$$1/\tilde{k}_0 = [(x-1)^4(3x+2) + 5x - 2]/6x(x-1)^4 \tag{19}$$

and

$$\tilde{k}_1 = [2(x-1)^5 - 2(5x-1)]/x^2(x-1)^3 \tag{20}$$

For $x = 0$, $1/\tilde{k}_0 = 0$ and $\tilde{k}_1 = 20$. For $x = 1$, $1/\tilde{k}_0 \rightarrow \infty$ and $\tilde{k}_1 \rightarrow \infty$. The envelope curve is shown in Fig. 2. Desirable values of $1/\tilde{k}_0$ and $\tilde{k}_1$ are on the inside curved side of the envelope curve.

2. Unity Condition

$$y^3 = h[x;\ 1/\tilde{k}_0,\ \tilde{k}_1] = 1 \quad 0 \leq x \leq 1 \tag{21}$$

The envelope in $1/\tilde{k}_0$ and $\tilde{k}_1$ with x as the variable param-

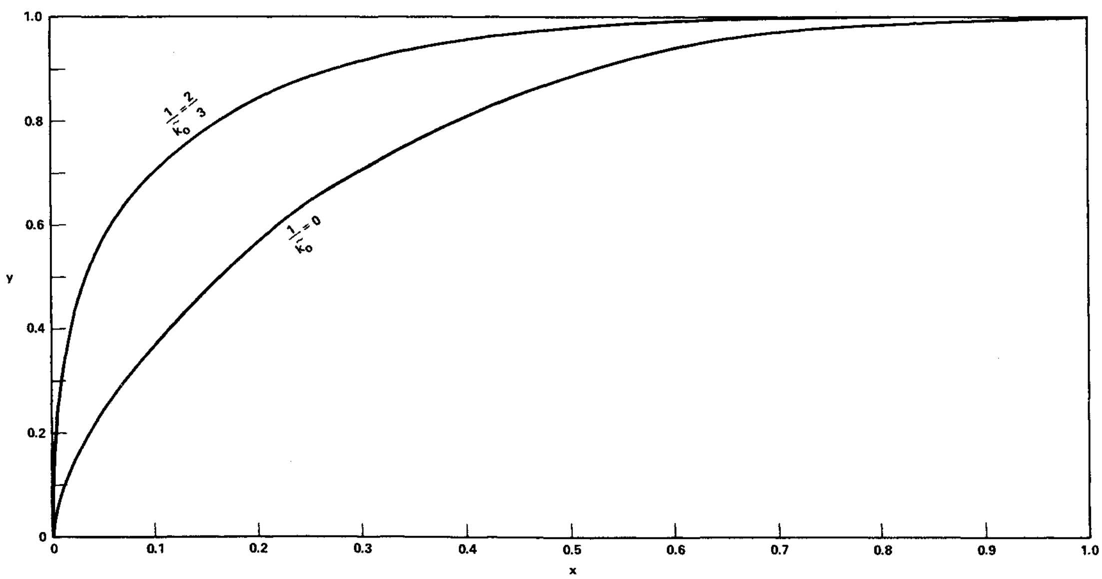

Fig. 3 One-parameter cubic polynomials.

eter is given by

$$\partial/\partial x(h-1) = \partial h/\partial x = 0 \tag{22}$$

The two envelope conditions, Eqs. (21) and (22), provide two simultaneous equations in $1/\tilde{k}_0$ and $\tilde{k}_1$ which are solved by the Cramer rule to yield

$$1/\tilde{k}_0 = (3x+2)/6x \tag{23}$$

and

$$\tilde{k}_1 = 2(x-1)^2/x^2 \tag{24}$$

For $x = 0$, $1/\tilde{k}_0 \rightarrow \infty$ and $\tilde{k}_1 \rightarrow \infty$. For $x = 1$, $1/\tilde{k}_0 = 5/6$ and $\tilde{k}_1 = 0$. The envelope curve is shown in Fig. 2 with desirable values of $1/\tilde{k}_0$ and $\tilde{k}_1$ on the inside curved side.

3. Maximum or Minimum Condition

$$\partial y/\partial x = h'[x;\ 1/\tilde{k}_0,\ \tilde{k}_1] = 0 \tag{25}$$

The envelope curve in $1/\tilde{k}_0$ and $\tilde{k}_1$ with x as the variable parameter is given by

$$h'' = 0 \tag{26}$$

The two envelope conditions, Eqs. (25) and (26), provide two simultaneous equations in $1/\tilde{k}_0$ and $\tilde{k}_1$ which are solved by the Cramer rule to produce

$$1/\tilde{k}_0 = 5x^2/(10^2 - 5x + 1) \tag{27}$$

and

$$\tilde{k}_1 = 20(x-1)^2/(10x^2 - 5x + 1) \tag{28}$$

For $x = 0$, $1/\tilde{k}_0 = 0$, $\tilde{k}_1 = 20$. For $x = 1$, $1/\tilde{k}_0 = 5/6$, $\tilde{k}_1 = 0$.

The envelope curve is shown in Fig. 2 with desirable values of $1/\tilde{k}_0$ and $\tilde{k}_1$ on the inside curved side.

4. Inflection Point Condition

For y^3 the inflection point condition becomes

$$3hh'' - 2(h')^2 = 0 \tag{29}$$

The envelope curve in $1/\tilde{k}_0$ and $\tilde{k}_1$ with x as the variable parameter is given by

$$3hh''' - h'h'' = 0 \tag{30}$$

These two conditions provide two simultaneous equations in $1/\tilde{k}_0$ and $\tilde{k}_1$ in terms of x. Since the equations are quadratic in $1/\tilde{k}_0$ and $\tilde{k}_1$, numerical procedures are necessary. The results are shown in Fig. 2.

Relative Fullness of Shape

The cubic polynomial as defined in normalized coordinates specifies an annular volume which is also governed by the relative diameter of the flat face. Consequently the prismatic coefficient is a function of not only the cubic polynomial parameters but also the relative diameter of the flat face.

A simple measure of the fullness of the volume specified by the cubic polynomial is given by the integral F where

$$F \equiv \int_0^1 y^3 dx \tag{31}$$

For Eq. (19)

$$F = 2/3 + (1/5)(1/\tilde{k}_0) - \tilde{k}_1/120 \tag{32}$$

For constant values of F, straight lines are indicated in Fig. 2. Fuller shapes are specified in the lower right-hand side of Fig. 2.

One-Parameter "Cubic" Polynomials

A one-parameter family of cubic polynomials may also be considered if $\tilde{k}_0$, say, is taken as the single adjustable parameter. The result is a fourth-degree polynomial or quartic. Then a_5 of Eq. (8) is zero and

$$\tilde{k}_1 = -12(1/\tilde{k}_0) + 8 \tag{33}$$

This is plotted in Fig. 2 as well as the values for the zero-parameter cubic polynomial, Eq. (1), $\tilde{k}_1 = 2$ and $1/\tilde{k}_0 = 1/2$.

By use of Eq. (33) in Eq. (9), the one-parameter cubic polynomial becomes

$$y^3 = (1/\tilde{k}_0)\tilde{K}_0[x] + Q[x] \tag{34}$$

where

$$\tilde{K}_0 = -6x(x-1)^3 \tag{35}$$

and

$$Q = x^2(3x^2 - 8x + 6) \tag{36}$$

A study of the permissible range of $1/\tilde{k}_0$ gives

$$0 \leq 1/\tilde{k}_0 \leq 2/3$$

to satisfy the zero, unity, maximum or minimum, and inflection point conditions. Representative curves are plotted in Fig. 3.

References

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